Hamiltonian varieties of universal algebras

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A group is called Hamiltonian if its every subgroup is normal. Generalizing this notion, T. EVANS [4] introduced the Hamiltonian property of loops. He called a loop **A** Hamiltonian if its every subloop is a block of certain congruence of **A**, and proved that a variety of loops is Hamiltonian (i.e., every loop is Hamiltonian in the variety) if and only if it is a variety of Abelian groups. In this paper we shall study a natural generalization of this notion for universal algebras and describe the Hamiltonian varieties of universal algebras. Our way is independent of that of Evans and furnishes, among others, a new proof of his result, too.

Definition. A universal algebra (shortly: algebra) $\langle A; \Omega\rangle$ is called *Hamiltonian* if its every subalgebra is a block of some congruence of $\langle A; \Omega\rangle$. A variety of algebras is called Hamiltonian if every algebra is Hamiltonian in it.

This condition was applied in some earlier papers of K. SHODA [6] and B. CSÁKÁNY [3].

We shall often use the following lemma which is a special case of a theorem of MAL'CEV ([5] Theorem 5).

Lemma. *Let $\langle A; \Omega\rangle$ be an algebra and B a subset of A. The subset B is a block of some congruence of $\langle A; \Omega\rangle$ if and only if for any translation τ of* **A** *either $B\tau \subseteq B$ or $B\tau \cap B = \emptyset$ ($B\tau$ is the image of B under the mapping τ).*

The Hamiltonian varieties are characterized by the following

Theorem 1. *A variety of algebras $\mathfrak{A}$ is Hamiltonian if and only if for any n-ary polynomial symbol g there exists a ternary polynomial symbol k_g such that the identity*

$$(*) \qquad g(x_1, x_2, \ldots, x_n) = k_g(x_0, g(x_0, x_2, \ldots, x_n), x_1)$$

holds in $\mathfrak{A}$.

Proof. Let the variety $\mathfrak{A}$ be Hamiltonian and consider the free algebra $\mathbf{F}(\in\mathfrak{A})$ generated by the set $\{x_0, x_1, \dots, x_n\}$. For any n-ary polynomial symbol g, the mapping $\tau: y \to g(y, x_2, \dots, x_n)$ $(y \in F)$ is a translation of $\mathbf{F}$. Consider the subalgebra $\mathbf{A}$ of $\mathbf{F}$ generated by the set $\{x_0, g(x_0, x_2, \dots, x_n), x_1\}$. Since $x_0\tau \in A$, by the lemma, we have $x_1\tau \in A$. Thus there exists a ternary polynomial symbol k_g such that the equality

$$x_1\tau = g(x_1, x_2, \dots, x_n) = k_g(x_0, g(x_0, x_2, \dots, x_n), x_1)$$

holds. Since $\mathbf{F}$ is a free algebra of the variety $\mathfrak{A}$ this equality is an identity in $\mathfrak{A}$.

Now, we suppose that for every n-ary polynomial symbol g of the variety $\mathfrak{A}$ there exists a ternary polynomial symbol k_g of $\mathfrak{A}$ satisfying the identity $(*)$. Let $\mathbf{D}(\in\mathfrak{A})$ be any algebra and $\mathbf{A}$ a subalgebra of $\mathbf{D}$. Consider a translation τ of $\mathbf{D}$ such that $a\tau \in A$ for some $a \in A$. This translation can be given by a polynomial symbol $g(x_1, x_2, \dots, x_n)$: $a\tau = g(a, d_2, \dots, d_n)$ where $d_2, \dots, d_n \in D$. Then, using $(*)$, for any element $b(\in A)$ we have

$$b\tau = g(b, d_2, \dots, d_n) = k_g(a, g(a, d_2, \dots, d_n), b)$$

and therefore $b\tau \in A$. By the lemma, the proof is complete.

In theorems 2—5 we establish some basic properties of Hamiltonian algebras.

Theorem 2. *For any algebra* $\mathbf{A}$ *the following two conditions are equivalent:*

(i) $\mathbf{A}$ *is Hamiltonian,*

(ii) *any subalgebra of* $\mathbf{A}$ *generated by three elements is a block of some congruence of* $\mathbf{A}$.

Proof. Since (i)$\Rightarrow$(ii) is obvious it is enough to prove that (ii)$\Rightarrow$(i). Suppose that every subalgebra of $\mathbf{A}$ generated by three elements is a block of certain congruence of $\mathbf{A}$ but $\mathbf{A}$ has such subalgebra $\mathbf{A}_1$ which is not a block of any congruence of $\mathbf{A}$. Thus, by the lemma, there are elements a_1, a_2, a_3 in A and $\mathbf{A}$ has a translation τ such that $a_1\tau = a_2$ and $a_3\tau \notin A_1$. Now the subalgebra of $\mathbf{A}$ generated by the set of three elements $\{a_1, a_2, a_3\}$ is not a block of any congruence in view of the lemma and we have got a contradiction, which completes the proof.

Theorem 3. *The Hamiltonian property is local.*

Proof. We shall show if an algebra $\mathbf{A}$ is not Hamiltonian then it has a finitely generated subalgebra which is not Hamiltonian.

Let $\mathbf{A}$ be any algebra and $\mathbf{A}_1$ a subalgebra of $\mathbf{A}$ which is not a block of any congruence of $\mathbf{A}$. Now, by the lemma, $\mathbf{A}$ has a translation τ $(a\tau = g(a, a_2, \dots, a_n)$ for every $a \in A)$ and A_1 has elements b, c, d such that

$$b\tau = c \quad \text{and} \quad d\tau \notin A_1.$$

Consider the subalgebra $\mathbf{A}_2$ of $\mathbf{A}$ generated by the finite set $\{b, c, d, a_2, \ldots, a_n\}$. The algebra $\mathbf{A}_1 \cap \mathbf{A}_2$ is such a subalgebra of $\mathbf{A}_2$ which is not a block of any congruence of $\mathbf{A}_2$ (τ is a translation of $\mathbf{A}_2$, too). The proof is complete.

The reader can prove in a routine way the next

Theorem 4. *Any homomorphic image and any subalgebra of a Hamiltonian algebra is Hamiltonian, too.*

In view of Birkhoff's characterization of varieties, by Theorems 3 and 4 the following question is raised: does the class of all Hamiltonian algebras of the same type form a variety? The answer is in the negative. Indeed, the Cartesian square of the quaternion group is not Hamiltonian.

Theorem 5. *In any category of Hamiltonian algebras the epimorphisms are surjections.*

Proof. Let $\mathbf{A}$ and $\mathbf{B}$ be Hamiltonian algebras. Suppose that the epimorphism $\varepsilon: \mathbf{A} \to \mathbf{B}$ is not surjective. Then $\mathbf{B}$ has a proper subalgebra $\mathbf{C}$ which is the image of $\mathbf{A}$ under the epimorphism ε. Since $\mathbf{B}$ is Hamiltonian, it has a congruence θ_c such that C is a block of θ_c. Consider the following homomorphisms α_1, α_2 from $\mathbf{B}$ into $\mathbf{B}/\theta_c$:

$$b\alpha_1 = [b]\theta_c, \quad b\alpha_2 = C$$

for all $b \in B$. Now, it is clear that $\varepsilon\alpha_1 = \varepsilon\alpha_2$. But $\alpha_1 \neq \alpha_2$ which is a contradiction.

Theorem 6. (T. Evans [4].) *A variety $\mathfrak{A}$ of loops is Hamiltonian if and only if it is a variety of Abelian groups.*

Proof. Let the variety $\mathfrak{A}$ of loops be Hamiltonian. Then $\mathfrak{A}$ is (polynomially) equivalent to a variety $\mathfrak{R}$ of all unital right $\mathbf{R}$-modules over a ring $\mathbf{R}$ with identity element (see [3], Theorem 4). Every operation of $\mathfrak{R}$ can be written as a sum of unary operations [2] and therefore we have the identity

$$(*) \qquad x \circ y = x\alpha + y\beta$$

where "$\circ$" is the operation of the loop and $\alpha, \beta \in R$. Under this equivalence the identity element e of the loop corresponds to the zero element of $\mathbf{R}$. Substituting $x=e$ and $y=e$ respectively we have got $\alpha=\beta=1$ (the identity element of $\mathbf{R}$). Thus the operation of the loop is associative and commutative.

In an earlier paper [7] we studied the Abelian property of universal algebras. We called the algebra $\langle A; \Omega\rangle$ Abelian, if for any operations μ and ν (m- and n-ary, erespctively) and any matrix $(a_{ij})_{nxm}$ over A the equality

$$(a_{11} \ldots a_{1m}\mu) \ldots (a_{n1} \ldots a_{nm}\mu)\nu = (a_{11} \ldots a_{n1}\nu) \ldots (a_{1m} \ldots a_{nm}\nu)\mu$$

holds. It is easy to verify that for varieties of loops, rings and lattices the Abelian and the Hamiltonian properties coincide, i.e., if a variety of loops (rings, lattices) is Hamiltonian then it is Abelian, and conversely. Therefore, a natural question can be raised: do the two properties coincide at any variety? The answer is in the negative. We can find a counter example among the varieties of unital semimodules over a semiring [2].

Theorem 7. *The variety $\mathfrak{R}$ of all unital right semimodules over a semiring* $\mathbf{R}$ *with identity element is Hamiltonian if and only if* $\mathbf{R}$ *is a ring.*

Proof. Let $\mathbf{F}\in\mathfrak{R}$ be a free algebra with the set of free generators $\{x_0, x_1, \dots, x_n\}$. Any polynomial symbol $g(x_1, x_2, \dots, x_n)$ of $\mathfrak{R}$ can be written in the form

$$g(x_1, \dots, x_n) = x_1\gamma_1 + \dots + x_n\gamma_n$$

where $\gamma_1, \dots, \gamma_n \in R$. If $\mathfrak{R}$ is Hamiltonian we have, by Theorem 1, ternary polynomial symbol $k_g(x, y, z) = x\varkappa_1 + y\varkappa_2 + z\varkappa_3$ such that in $\mathbf{F}$ the equality

$$x_1\gamma_1 + x_2\gamma_2 + \dots + x_n\gamma_n = x_0\varkappa_1 + (x_0\gamma_1 + x_2\gamma_2 + \dots + x_n\gamma_n)\varkappa_2 + x_1\varkappa_3$$

holds. Since $\mathbf{F}$ is a free algebra in $\mathfrak{R}$ this equality is an identity in $\mathfrak{R}$ and we have the following set of equations

$$(*) \qquad \varkappa_1 + \gamma_1\varkappa_2 = 0, \quad \varkappa_3 = \gamma_1, \quad \gamma_i\varkappa_2 = \gamma_i \quad (i = 2, 3, \dots, n).$$

This set of equations has to be valid for any polynomial symbol g with suitable $\varkappa_1$, $\varkappa_2$ and $\varkappa_3$. If $\gamma_i = 1$ for $i = 1, \dots, n$ we have got from $(*)$ that $\varkappa_2 = \varkappa_3 = 1$ and $\varkappa_1 + \varkappa_2 = 0$, hence it follows $\varkappa_1 = -1$, therefore $\mathbf{R}$ is a ring.

On the other hand, if $\mathbf{R}$ is a ring then the set of equations $(*)$ can be solved at any γ_i $(i = 1, \dots, n)$. Indeed, $\varkappa_1 = -\gamma_1$, $\varkappa_2 = 1$ and $\varkappa_3 = \gamma_1$ are solutions.

Theorem 8. *The variety $\mathfrak{R}$ of all unital right semimodules over a semiring* $\mathbf{R}$ *with identity element is Abelian if and only if* $\mathbf{R}$ *is a commutative semiring.*

Proof. Let the variety $\mathfrak{R}$ be Abelian and consider the following operation "$\circ$" of $\mathfrak{R}$:

$$x_1 \circ x_2 = x_1\alpha + x_2\beta,$$

where $\alpha, \beta \in R$. Since the semiring $\mathbf{R}$ under its own addition and right multiplication is a unital $\mathbf{R}$-semimodule in $\mathfrak{R}$, by the definition of the Abelian property, we have $\alpha\beta = \beta\alpha$, using the matrix

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Such operations can be constructed by any elements α, β of R^{*} and thus we have proved that the multiplication in **R** is commutative.

The sufficiency is obvious.

Since the classes of rings and commutative semirings do not contain each other there exist Abelian and non Hamiltonian varieties and conversely.

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